Assignment 2: Triple Integrals: Solutions

1. (a) By Fubini's theorem, we can evaluate the integral in any order we want. Integrating z first, then y, then x:

$$\begin{aligned}
& \int_1^2 \int_{-1}^0 \int_0^3 x + 2y + 4z \, dz \, dy \, dx \\
= & \int_1^2 \int_{-1}^0 (x+2y)z + 2z^2|_0^3 \, dy \, dx \\
= & \int_1^2 \int_{-1}^0 3x + 6y + 18 \, dy \, dx \\
= & \int_1^2 (3x+18)y + 3y^2|_{-1}^0 \, dx \\
= & \int_1^2 3x + 18 - 3 \, dx \\
= & \frac{3}{2}x^2 + 15x|_1^2 \\
= & (6+30) - (\frac{3}{2} + 15) \\
= & \frac{39}{2}
\end{aligned}$$

(b)

$$\begin{aligned}
& \int_{-1}^2 \int_1^{x^2} \int_0^{x+y} (2x^2y) \, dz \, dy \, dx \\
= & \int_{-1}^2 \int_1^{x^2} (2x^2y)(x+y) \, dy \, dx \\
= & \int_{-1}^2 \int_1^{x^2} 2x^3y + 2x^2y^2 \, dy \, dx \\
= & \int_{-1}^2 x^3y^2 + \frac{2}{3}x^2y^3|_1^{x^2} \, dx \\
= & \int_{-1}^2 x^7 + \frac{2}{3}x^8 - x^3 - \frac{2}{3}x^2 \, dx \\
= & \frac{x^8}{8} + \frac{2x^9}{27} - \frac{x^4}{4} - \frac{2x^3}{9}|_{-1}^2 \\
= & 2^5 + \frac{2^{10}}{27} - 4 - \frac{16}{9} - \frac{1}{8} + \frac{2}{27} + \frac{1}{4} - \frac{2}{9} \\
= & \frac{513}{8}
\end{aligned}$$

(c) First, we need to find boundary equations for E. In the xy-plane, the solid is bounded by the lines $x+y=1$, $x=0$, $y=0$. This can be described as the region bounded by $0 \leq x \leq 1$, $0 \leq y \leq 1-x$.

Above this, in the z plane, lies the plane which contains the points (1,0,0), (0,1,0), and (0,0,2). By inspection, or by using the method described in section 12.5 of the textbook, we can find that the equation of this plane is $x+y+\frac{z}{2}=1$. Re-arranging for z gives $z=2-2x-2y$. Thus, our integral is

$$\begin{aligned}
& \int_0^1 \int_0^{1-x} \int_0^{2-2x-2y} (xy)\, dz\, dy\, dx \\
= & \int_0^1 \int_0^{1-x} (xy)(2-2x-2y)\, dy\, dx \\
= & \int_0^1 \int_0^{1-x} 2xy - 2x^2y - 2xy^2\, dy\, dx \\
= & \int_0^1 xy^2 - x^2y^2 - \frac{2}{3}xy^3|_0^{1-x}\, dx \\
= & \int_0^1 x(1-x)^2 - x^2(1-x)^2 - \frac{2}{3}x(1-x)^3\, dx \\
= & \int_0^1 x(1-2x+x^2) - x^2(1-2x+x^2) - \frac{2}{3}x(1-3x+3x^2-x^3)\, dx \\
= & \int_0^1 x - 2x^2 + x^3 - x^2 + 2x^3 - x^4 - \frac{2}{3}x + 2x^2 - 2x^3 + \frac{2}{3}x^4\, dx \\
= & \int_0^1 -\frac{1}{3}x^4 + x^3 - x^2 + \frac{1}{3}x\, dx \\
= & -\frac{1}{15}x^5 + \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{6}x^2|_0^1 \\
= & -\frac{1}{15} + \frac{1}{4} - \frac{1}{3} + \frac{1}{6} \\
= & \frac{1}{60}
\end{aligned}$$

2. First, we need to determine the region E. In the xz plane, the projection of E is the intersection of the curves $z=x^2$ and $z=x^3$. Since these curves intersect at $x=0$ and $x=1$, this region can be described as $0 \leq x \leq 1$, $x^3 \leq z \leq x^2$.

Furthermore, the bounds for y are given by the equations $y = z^2$ and $y = 0$. Thus, the volume is given by

$$\begin{aligned}
& \int_0^1 \int_{x^3}^{x^2} \int_0^{z^2} 1\, dy\, dz\, dx \\
= & \int_0^1 \int_{x^3}^{x^2} z^2\, dy\, dx \\
= & \int_0^1 \frac{z^3}{3} |_{x^3}^{x^2}\, dx \\
= & \int_0^1 \frac{x^6}{3} - \frac{x^9}{3}\, dx \\
= & \frac{x^7}{21} - \frac{x^{10}}{30} |_0^1 \\
= & \frac{1}{21} - \frac{1}{30} \\
= & \frac{1}{70}
\end{aligned}$$

3. Again, we first need to determine the bounds of the solid. In the xz-plane (below, we give the setup for projecting into the yz-plane), the projection of the solid is given by the curves $z+x^2 = 4$ and $z = 0$. These intersect at $x = \pm 2$, so the bounds are $-2 \leq x \leq 2$ and $0 \leq z \leq 4 - x^2$.

   The bounds for y are given by $y = 4 - z$, $y = 0$. Thus the total mass is

$$\int_{-2}^2 \int_0^{4-x^2} \int_0^{4-z} m(x,y,z)\, dy\, dz\, dx$$

   But the mass is constantly 5, so $m(x, y, z) = 5$. Thus the above is

$$\begin{aligned}
= & \int_{-2}^2 \int_0^{4-x^2} \int_0^{4-z} 5\, dy\, dz\, dx \\
= & 5 \int_{-2}^2 \int_0^{4-x^2} 4 - z\, dz\, dx \\
= & 5 \int_{-2}^2 4z - \frac{z^2}{2} |_0^{4-x^2}\, dx \\
= & 5 \int_{-2}^2 16 - 4x^2 - 8 + 4x^2 - \frac{x^4}{2}\, dx
\end{aligned}$$

$$\begin{aligned}
&= 5\int_{-2}^{2} 8 - \frac{x^4}{2}\,dx \\
&= 5\left(16 - \frac{32}{10}\right) - \left(-16 + \frac{32}{10}\right) \\
&= 128
\end{aligned}$$

Alternatively, one could consider the projection in the yz-plane. Here, the curves are $y + z = 4$, $y = 0$, $z = 0$. This gives bounds $0 \leq z \leq 4$ and $0 \leq y \leq 4 - z$. The bounds for x are then given by re-arranging $z + x^2 + 4$ and solving for x: $-\sqrt{4-z} \leq x \leq \sqrt{4-z}$, and so one could also evaluate the integral

$$\int_0^4 \int_0^{4-z} \int_{-\sqrt{4-z}}^{\sqrt{4-z}} 5\,dx\,dy\,dz$$

Evaluating this integral also gives 128.

4. Since the region is a hemisphere, this will be easiest to solve if we re-write it in spherical co-ordinates. Since $z \geq 0$, the hemisphere has bounds $0 \leq \phi \leq \frac{\pi}{2}$, $0 \leq \theta \leq 2\pi$, and $0 \leq r \leq 1$. Since $x^2 + y^2 + z^2 = r^2$, the integral reduces to

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 (r)(r^2 \sin\phi)\,dr\,d\theta\,d\phi \\
&= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \frac{r^4}{4} \sin\phi|_0^1\,d\theta\,d\phi \\
&= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \frac{\sin\phi}{4}\,d\theta\,d\phi \\
&= \int_0^{\frac{\pi}{2}} \frac{2\pi}{4} \sin\phi\,d\phi \\
&= \frac{\pi}{2}(-\cos\phi)|_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{2}(0 - (-1)) \\
&= \frac{\pi}{2}
\end{aligned}$$

5. This is a straightforward calculation: the Jacobian reduces to

$$2(4) - 3(5) - 1(-1) = -6$$

6. First, we need to solve the equations for x and y. After re-arranging the equations, we find

$$x = \frac{1}{2}u + \frac{1}{2}v \text{ and } y = -\frac{1}{2}u + \frac{1}{2}v$$

The Jacobian of this transformation can be calculated as $\frac{1}{2}$.

Next, we need to change the bounds of the integral. Under the transformation, the four points of the diamond change to the four points $(-1, 1), (-1, 3), (1, 3), (1, 1)$. Thus the new region is the rectangle $-1 \leq u \leq 1$, $1 \leq v \leq 3$. Thus the integral becomes

$$\begin{aligned}
& \int_{-1}^{1} \int_{1}^{3} u^2 \cos^2 v \frac{1}{2}\, dv\, du \\
= & \frac{1}{2} \int_{-1}^{1} u^2 \int_{1}^{3} \frac{1 + \cos(2v)}{2}\, dv\, du \\
= & \frac{1}{4} \left(\frac{u^3}{3} |_{-1}^{1} \cdot \left(\frac{\sin(2v)}{2} + v \right) |_{1}^{3} \right) \\
= & \frac{1}{4} \left(\frac{2}{3} \cdot \left(\frac{\sin 6}{2} + 3 - \frac{\sin 2}{2} - 1 \right) \right) \\
= & \frac{1}{6} \left(\frac{\sin 6}{2} - \frac{\sin 2}{2} + 2 \right)
\end{aligned}$$